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A Priori Bounds and Upper and Lower Solutions for Nonlinear Second-Order Boundary-Value Problems

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1. INTRODUCTION

We obtain *a priori* bounds for solutions to

$$y'' = f(t, y, y'), \tag{1.1}$$

$$y(a) - g_1(y'(a)) = 0, \tag{1.2}$$

$$y(b) + g_2(y'(b)) = 0, \tag{1.3}$$

where we assume throughout that $f(t, y, y')$ is continuous on $[a, b] \times R^2$, and $g_1(y')$ and $g_2(y')$ are continuous and nondecreasing in $(-\infty, \infty)$.

Our method originates from the observation that if $y(t)$ is a solution to (1.1)–(1.3) with $\max_{[a,b]} y(t) = y(t_0) = M \geqslant 0$ and $a < t_0 < b$, then $y(t)$ is a solution to

$$y'' = f(t, y, y'), \tag{1.4}$$

$$y(t_0) = M, \tag{1.5}$$

$$y'(t_0) = 0. \tag{1.6}$$

Solutions to (1.4)–(1.6) are then compared with solutions to an auxilliary problem

$$x'' = -\psi(x, x'), \tag{1.7}$$

$$x(t_0) = M, \tag{1.8}$$

$$x'(t_0) = 0, \tag{1.9}$$

where $f(t, y, y') > -\psi(y, y')$ on an appropriate set and

(A) The function $\psi(x, x')$ is positive, continuous, and satisfies a local Lipschitz condition on R^2.

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In order to make use of first-order differential inequalities the problem (1.7)–(1.9) is reduced to the sequence of first-order problems

$$dh/d\sigma = -\sigma/\psi(h, \sigma), \tag{1.10}$$

$$h(0) = M \geqslant 0, \tag{1.11}$$

$$d\sigma/dt = -\psi(h(\sigma), \sigma), \tag{1.12}$$

$$\sigma(t_0) = 0. \tag{1.13}$$

We denote the unique solution to problem (1.10), (1.11) for $M \geqslant 0$ by $h(\sigma, M)$ and its maximal interval of existence by $(W_-(M), W_+(M))$. We denote the unique solution to (1.7)–(1.9) [equivalently, (1.12), (1.13)] by $x(t, t_0, M)$.

In Section 2 we prove the central lemma of this paper which gives an *a priori* upper bound on solutions to (1.1)–(1.3) under the following conditions in addition to (A):

(B) There exists $M_1 \geqslant 0$ such that if $M \geqslant M_1$,

$$\int_{W_-(M)}^{0} d\sigma/\psi(h(\sigma, M), \sigma) > b - a$$

$$\left(\int_0^{W_+(M)} d\sigma/\psi(h(\sigma, M), \sigma) > b - a\right).$$

(C) There exists $M_2 \geqslant 0$ such that for $M \geqslant M_2$

$$f(t, y, y') > -\psi(y, y')$$

on

$$\{(t, y, y') : x'(b, a, M) \leqslant y' \leqslant 0, y \geqslant h(y', M)\}$$

$$(\{(t, y, y') : 0 \leqslant y' \leqslant x'(a, b, M), y \geqslant h(y', M)\}).$$

(D) There exists $M_3 \geqslant 0$ such that if $M \geqslant M_3$,

$$x(b, a, M) + g_2(x'(b, a, M)) > 0$$

$$(x(a, b, M) - g_1(x'(a, b, M)) > 0).$$

Analogous hypotheses yield a lower bound. Hypothesis (B) is used to obtain a global existence lemma (Lemma 2.2), and hypothesis (C) plays the crucial role in a comparison lemma (Lemma 2.4).

In Section 3 we obtain several *a priori* bound theorems which are consequences of the central lemma. These are obtained by estimating or calculating $h(\sigma, M)$ and $x(t, t_0, M)$ whenever $\psi(x, x')$ takes particular forms. One of these theorems is an improvement of a theorem of the author [1].

Other conditions for *a priori* bounds for solutions to (1.1)–(1.3) may be found in the paper of Klokov [2].

In Section 4 we show that the hypotheses of any of the theorems of Section 3 imply that $x(t, a, M)$ $(x(t, b, M))$ for M sufficiently large is an upper solution in the sense of the theory developed by Jackson and others (see, for example, [3–6]). We then show that each of the theorems of Section 3 has a corresponding existence theorem which may be obtained either using the Leray–Schauder fixed-point theorem or using the existence theorems of Erbe [4] and Bebernes and Fraker [5]. Though the existence theorems obtainable by means of this paper are less general than those in [4] and [5], their hypotheses give more explicit conditions on $f(t, y, y')$. Among these theorems are improvements of the existence portions of the theorems of Bebernes and the author [7], and Waltman [8].

2. The Central Lemma

We first show that (1.7)–(1.9) is equivalent to (1.10)–(1.13). This equivalence is established by the following lemma.

Lemma 2.1. *Let $\psi(\rho, \sigma)$ satisfy condition* (A). *Then*

$$x(t, t_0, M) = h(x'(t, t_0, M), M)$$

on the maximal interval of existence of $x(t, t_0, M)$.

Proof. For simplicity of notation let $x(t) \equiv x(t, t_0, M)$. We have $(x')'(t) = -\psi(x(t), x'(t)) < 0$, thus $\sigma = x'(t)$ has a continuously differentiable inverse $t = r(\sigma)$. Define $H(\sigma) \equiv x(r(\sigma))$. We have

$$\begin{aligned} dH/d\sigma &= (dx/dt)(dt/d\sigma) \\ &= (dx/dt)/(d\sigma/dt) \\ &= \sigma/-\psi(H(\sigma), \sigma). \end{aligned}$$

Moreover, $H(0) = M$. Thus, by uniqueness, $H(\sigma) = h(\sigma, M)$; i.e., $x(t) = h(x'(t), M)$.

Remark 1. Note that if $x(t)$ is a solution to (1.7)–(1.9), then $x'(t)$ is a solution to (1.12), (1.13) with $h(\sigma) = h(\sigma, M)$. Conversely, if $\sigma(t)$ is a solution to (1.12), (1.13) with $h(\sigma) = h(\sigma, M)$, then $x(t) \equiv h(\sigma(t), M)$ is a solution to (1.7)–(1.9). It is the identity $x(t, t_0, M) \equiv h(x'(t, t_0, M), M)$ which we use in the sequel.

Remark 2. Note that $h(\sigma, M)$ is defined for $\{\sigma : \sigma = x'(t, t_0, M)$ for some t and $t_0\}$. In particular, $W_-(M) < x'(b, a, M) < x'(a, b, M) < W_+(M)$ when the solutions $x(t, a, M)$ and $x(t, b, M)$ extend to $[a, b]$.

LEMMA 2.2. (Global Existence). *If conditions* (A) *and* (B) *are satisfied, then for* $M \geqslant M_1$ *the solution* $x(t, t_0, M)$ *to* (1.7)–(1.9) *exists on* $[t_0, b]([a, t_0])$.

Proof. For simplicity, let $x(t) \equiv x(t, t_0, M)$. We have

$$\begin{aligned} x''(t) &= -\psi(x(t), x'(t)) \\ &= -\psi(h(x'(t), M), x'(t)). \end{aligned}$$

Thus

$$x''(t)/\psi(h(x'(t), M), x'(t)) = -1$$

$$\int_{t_0}^{t} x''(s)\, ds/\psi(h(x'(s), M), x'(s)) = -(t - t_0)$$

$$\int_0^{x'(t)} d\sigma/\psi(h(\sigma, M), \sigma) = -(t - t_0).$$

Suppose the maximal interval of existence to the right is $[t_0, w)$, where $w \leqslant b$, then $x'(t) \to -\infty$ and by the preceding remark $W_-(M) = -\infty$. Thus

$$\int_{-\infty}^{0} d\sigma/\psi(h(\sigma, M), \sigma) = w - t_0 \leqslant b - a.$$

This contradicts (B).

Remark. The converse of Lemma 2.2 is true in the following sense. If $x(t, a, M)$ (or $x(t, b, M)$) is defined on $[a, b]$, then

$$\int_{x'(b,a,M)}^{0} \frac{d\sigma}{\psi(h(\sigma, M), \sigma)} = b - a$$

and

$$\int_{W_-(M)}^{0} d\sigma/\psi(h(\sigma, M), \sigma) > b - a.$$

LEMMA 2.3. *Suppose conditions* (A) *and* (B) *are satisfied and* $M \geqq M_1$. *Suppose* $y(t) \in C^2[t_0, t_1]$ $(C^2[t_1, t_0])$, $y(t_0) = M$, $y'(t_0) = 0$, *and*

$$y''(t) > -\psi(y(t), y'(t)) \quad \text{on} \quad \{t : y(t) \geqslant h(y'(t), M), x'(b, a, M) \leqslant y'(t) \leqslant 0\}$$

$(\{t : y(t) \geqslant h(y'(t), M), 0 \leqslant y'(t) \leqslant x'(a, b, M)\})$. *If* $y(t) \leqslant M$, $y'(t_1) \leqslant 0$ $(y'(t_1) \geqslant 0)$, *and* $x'(b, a, M) \leqslant y'(t)$ *on* $[t_0, t_1]$ $(y'(t) \leqslant x'(a, b, M)$ *on* $[t_1, t_0])$, *then* $y(t_1) \leqslant h(y'(t_1), M)$.

Proof. Suppose $y(t_1) > h(y'(t_1), M)$. Define

$$t_2 \equiv \inf\{\tau : y(t) > h(y'(t), M) \text{ on } (\tau, t_1]\}.$$

Note that $t_2 < t_1$, $y(t_2) = h(y'(t_2), M)$, and $x'(b, a, M) \leqslant y'(t) \leqslant 0$ on $[t_2, t_1]$. This follows since whenever $y'(t) = 0$, $h(y'(t), M) = M \geqslant y(t)$. It then follows that $y(t) > h(y'(t), M)$ on $(t_2, t_1]$. We must have one of the following three cases.

Case 1: $y''(t_2) > 0$. Then $\sigma = y'(t)$ has an increasing continuously differentiable inverse $t = r(\sigma)$ defined on an interval $y'(t_2) \leqslant \sigma < y'(t_2 + \alpha)$ where $t_2 + \alpha < t_1$ and $y''(r(\sigma)) > 0$ on $y'(t_2) \leqslant \sigma < y'(t_2 + \alpha)$. Define $H(\sigma) \equiv y(r(\sigma))$. Then

$$\begin{aligned} dH/d\sigma &= (dy/dt)(dt/d\sigma) \\ &= (dy/dt)/(d\sigma/dt) \\ &\leqslant 0 \leqslant \sigma/(-\psi(H(\sigma), \sigma)) \end{aligned}$$

for $\sigma \in [y'(t_2), y'(t_2 + \alpha))$. Moreover, $H(y'(t_2)) = h(y'(t_2), M)$. Thus, by a well-known theorem on first-order differential inequalities [10, p. 26], $H(\sigma) \leqslant h(\sigma, M)$ on $[y'(t_2), y'(t_2 + \alpha))$. But then $y(t) \leqslant h(y'(t), M)$ on $[t_2, t_2 + \alpha)$. This contradicts the definition of t_2.

Case 2: $y''(t_2) < 0$. The argument is similar to that in case 1.

Case 3: $y''(t_2) = 0$. Note that

$$\min_{[t_2, t_1]} [y''(t) + \psi(y(t), y'(t))] > 0.$$

Let $\delta > 0$ be chosen sufficiently small so that if $y_\delta(t) \equiv y(t) - \delta(t - t_2)^2$, then $y_\delta''(t) > -\psi(y_\delta(t), y_\delta'(t))$ on $[t_2, t_1]$, and $y_\delta(t_1) > h(y_\delta'(t_1))$. Note that $y_\delta(t_2) = h(y_\delta'(t_2))$.

Let

$$t_3 \equiv \inf\{\tau : y_\delta(t) > h(y_\delta'(t), M) \text{ on } (\tau, t_1]\}.$$

Note that $[t_3, t_1] \subset [t_2, t_1]$ and $y_\delta(t_3) = h(y_\delta'(t_3), M)$. If $y_\delta''(t_3) > 0$ or $y_\delta''(t_3) < 0$ we may argue as above to reach a contradiction; hence, we assume $y_\delta''(t_3) = 0$. We have

$$d(h(y_\delta'(t), M))/dt\,|_{t=t_3} = (dh/d\sigma)\, y_\delta''(t)|_{t=t_3} = 0$$

and

$$d(y_\delta(t))/dt\,|_{t=t_3} = y'(t_3) - 2\delta(t_3 - t_2) < 0.$$

The latter inequality follows since $y'(t) \leqslant 0$ on $[t_3, t_1]$, and $t_2 < t_3$ since $y''_\delta(t_2) = -2\delta < 0$. Thus $y_\delta(t) \leqslant h(y_\delta'(t), M)$ on some interval $[t_3, t_3 + \alpha)$. This contradicts the definition of t_3.

LEMMA 2.4 (Comparison). *Let conditions* (A), (B), *and* (C) *be satisfied. Let $y(t)$ be a solution to*

$$y'' = f(t, y, y'),$$
$$y(t_0) = M,$$
$$y'(t_0) = 0,$$

with maximal interval of existence (w_-, w_+) such that $y(t) \leqslant M$ and $M \geqslant M_1, M_2$. Then $y(t) \geqslant x(t, t_0, M)$ and $y'(t) \geqslant x'(t, t_0, M)$ ($y'(t) \leqslant x'(t, t_0, M)$) on $[t_0, b] \cap [t_0, w_+)$ ($[a, t_0] \cap (w_-, t_0]$).

Proof. For simplicity of notation, let $x(t) \equiv x(t, t_0, M)$. Let

$$t_1 \equiv \sup\{\tau : x(t) \leqslant y(t) \text{ and } x'(t) \leqslant y'(t) \text{ for } t_0 \leqslant t \leqslant \tau\}.$$

Suppose that $t_1 < \min[b, w_+]$. We must have one of the following cases:

Case 1: $x(t_1) < y(t_1)$, $x'(t_1) = y'(t_1)$. Note that $y(t) \leqslant M$ on $[t_0, t_1]$, $y'(t_1) \leqslant 0, y'(t) \geqslant x'(t, t_0, M) \geqslant x'(b, a, M)$ on $[t_0, t_1]$. By Lemmas 2.1 and 2.3, $x(t_1) = h(x'(t_1), M) = h(y'(t_1), M) \geqslant y(t_1)$. This is a contradiction.

Case 2: $x(t_1) = y(t_1)$, $x'(t_1) < y'(t_1)$. Then $x(t) \leqslant y(t)$ and $x'(t) \leqslant y'(t)$ on an interval $[t_1, t_1 + \beta]$. This contradicts the definition of t_1.

Case 3: $x(t_1) = y(t_1)$, $x'(t_1) = y'(t_1)$. Then

$$\begin{aligned} y''(t_1) = f(t_1, y(t_1), y'(t_1)) &> -\psi(y(t_1), y'(t_1)) \\ &> -\psi(x(t_1), x'(t_1)) = x''(t_1). \end{aligned}$$

Then $y'(t) \geqslant x'(t)$ and hence $y(t) \geqslant x(t)$ in a neighborhood $[t_1, t_1 + \beta)$. This contradicts the definition of t_1.

Remark. Note that in Lemmas 2.3 and 2.4, hypothesis (B) was used only indirectly to guarantee the existence of $x(b, a, M)$ and $x'(b, a, M)$.

LEMMA 2.5.1 (Central Lemma). *Suppose conditions* (A), (B), (C) *and* (D) *are satisfied. If $y(t)$ is a solution to* (1.1)–(1.3), *then*

$$y(t) \leqslant R \equiv \max[M_1, M_2, M_3, g_1(0), -g_2(0)].$$

Proof. Suppose for contradiction that $\max_{[a,b]} y(t) = y(t_0) = M > R$. If $t_0 = a$, then $y'(a) \leqslant 0$, and we have $y(a) - g_1(y'(a)) > 0$ which contradicts the fact that $y(t)$ satisfies (1.2). Similarly, if $t_0 = b$, $y(b) + g_2(y'(b)) > 0$. Thus we may assume $t_0 \in (a, b)$.

We then have that $y(t)$ is a solution to (1.4)–(1.6) with $y(t) \leqslant M$. By Lemma 2.4, $y(t) \geqslant x(t, t_0, M)$ and $y'(t) \geqslant x'(t, t_0, M)$ on $[t_0, b]$. Thus $y(b) \geqslant x(b, t_0, M)$, $y'(b) \geqslant x'(b, t_0, M)$.

We observe that $x(t, a, M) \equiv x(t + (t_0 - a), t_0, M)$. Since $x(t, t_0, M)$ and $x'(t, t_0, M)$ are both decreasing for $t \geqslant t_0$, we have $y(b) \geqslant x(b, t_0, M) \geqslant x(b, a, M)$, and $y'(b) \geqslant x'(b, t_0, M) \geqslant x'(b, a, M)$. But then by hypothesis (D),

$$y(b) + g_2(y'(b)) \geqslant x(b, a, M) + g_2(x'(b, a, M)) > 0$$

which contradicts the fact that $y(t)$ satisfies (1.3).

For reference in ensuing sections we state the analogous result for lower bounds. The proof is similar to that of Lemma 2.5.1.

LEMMA 2.5.2. *Suppose condition* (A) *is satisfied. For* $M < 0$, *let* $h(\sigma, M)$ *denote the unique solution to*

$$dh/d\sigma = \sigma/\psi(h, \sigma), \tag{2.1}$$

$$h(0) = M, \tag{2.2}$$

with maximal interval of existence $(W_-(M), W_+(M))$. *For* $M < 0$ *let* $x(t, t_0, M)$ *denote the unique solution to*

$$x'' = \psi(x, x'), \tag{2.3}$$

$$x(t_0) = M, \tag{2.4}$$

$$x'(t_0) = 0. \tag{2.5}$$

Assume

(B′) *There exists* $M_1 > 0$ *such that if* $M \leqslant -M_1$,

$$\int_{W_-(M)}^{0} d\sigma/\psi(h(\sigma, M), \sigma) > b - a$$

$$\left(\int_{0}^{W_+(M)} d\sigma/\psi(h(\sigma, M), \sigma) > b - a\right).$$

(C′) *There exists* $M_2 > 0$ *such that if* $M \leqslant -M_2$,

$$f(t, y, y') < \psi(y, y')$$

on

$$\{(t, y, y') : x'(a, b, M) \leqslant y' \leqslant 0, y \leqslant h(y', M)\}$$
$$(\{(t, y, y') : 0 \leqslant y' \leqslant x'(b, a, M), y \leqslant h(y', M)\}).$$

(D′) *There exists* $M_3 > 0$ *such that if* $M \leqslant -M_3$,

$$x(a, b, M) - g_1(x'(a, b, M)) < 0$$
$$(x(b, a, M) + g_2(x'(b, a, M)) < 0).$$

If $y(t)$ *is a solution to* (1.1)–(1.3), *then*

$$y(t) \geqslant \min[-M_1, -M_2, -M_3, g_1(0), -g_2(0)].$$

3. A Priori Bound Theorems

In this section we use the central lemma to obtain more explicit sufficient conditions for the existence of *a priori* bounds. These are obtained by calculating or estimating $h(\sigma, M)$ and $x(t, t_0, M)$.

Theorem 3.1. *Suppose*

(a) $\phi(\rho, \sigma)$ *is positive, continuous, satisfies a local Lipschitz condition in* ρ *and* σ, *and is nondecreasing in* ρ *and* σ *on* $[0, +\infty) \times [0, +\infty)$.

(b) $\int_0^\infty d\sigma/\phi(M, \sigma) > b - a$ *for all* $M \geqslant 0$.

(c) *There exists* $M_2 \geqslant 0$ *such that for* $M \geqslant M_2$,

$$f(t, y, y') > -\phi(|y|, |y'|)$$

on

$$\{(t, y, y') : -B(M) \leqslant y' \leqslant 0, y \geqslant L(y', M)\}$$
$$(\{(t, y, y') : 0 \leqslant y' \leqslant B(M), y \geqslant L(y', M)\}),$$

where $B(M)$ *is defined by*

$$\int_0^{B(M)} d\sigma/\phi(M, \sigma) = b - a,$$

and $L(\sigma, M)$ *is defined by*

$$\int_{L(\sigma, M)}^{M} \phi(|u|, 0)\, du = \sigma^2/2.$$

(d) *There exists* $M_3 \geqslant 0$ *such that for* $M \geqslant M_3$,

$$L(-B(M), M) + g_2(-B(M)) > 0$$
$$(L(B(M), M) - g_1(B(M)) > 0).$$

If $y(t)$ *is a solution to* (1.1)–(1.3), *then* $y(t) \leqslant \max[M_2, M_3, g_1(0), | g_2(0)|]$ $(\max[M_2, M_3, | g_1(0)|, -g_2(0)])$.

Proof. We apply Lemma 2.5.1 with $\psi(x, x') = \phi(| x |, | x' |)$. It is easily seen that (A) is satisfied. If we let $h(\sigma, M)$ denote the solution to

$$dh/d\sigma = -\sigma/\phi(| h(\sigma)|, | \sigma |),$$
$$h(0) = M,$$

then

$$\phi(| h(\sigma, M)|, | \sigma |) \, dh/d\sigma = -\sigma.$$

For $\sigma \leqslant 0$, since ϕ is nondecreasing in $| \sigma |$,

$$\phi(| h(\sigma, M)|, 0) \, dh/d\sigma \leqslant -\sigma.$$

Thus

$$\int_\sigma^0 \phi(| h(s, M)|, 0)(dh(s, M)/d\sigma) \, ds \leqslant \sigma^2/2.$$

Making the substitution $u = h(s, M)$, we obtain

$$\int_{h(\sigma,M)}^M \phi(| u |, 0) \, du \leqslant \sigma^2/2.$$

For $\sigma \geqslant 0$, we obtain the same inequality by a similar argument. We may conclude that $h(\sigma, M) \geqslant L(\sigma, M)$. Note that since $\phi(| u |, 0) \geqslant \phi(0, 0)$, $(W_-(M), W_+(M)) = (-\infty, \infty)$.

We observe that for $M \geqslant M_3$,

$$L(-B(M), M) > -g_2(-B(M)) \geqslant -g_2(0).$$

Thus for $M \geqq \max\{M_3, | g_2(0)|\}$, $L(-B(M), M) \geqslant -M$. Thus,

$$-M \leqslant L(\sigma, M) \leqslant h(\sigma, M) \leqslant M \qquad \text{for} \quad -B(M) \leqslant \sigma \leqslant 0.$$

Since $\phi(\rho, \sigma)$ is nondecreasing in ρ,

$$\int_{-\infty}^0 d\sigma/\phi(| h(\sigma, M)|, | \sigma |) > \int_{-B(M)}^0 d\sigma/\phi(M, | \sigma |)$$
$$= \int_0^{B(M)} d\sigma/\phi(M, \sigma) = b - a.$$

Thus condition (B) is satisfied.

By the remark following Lemma 2.2,

$$\int_{x'(b,a,M)}^{0} d\sigma/\phi(|\,h(\sigma, M)|, |\,\sigma\,|) = b - a.$$

It follows that $-B(M) \leqslant x'(b, a, M)$. Thus condition (C) is satisfied.
Moreover,

$$x(b, a, M) + g_2(x'(b, a, M)) \geqslant L(-B(M), M) + g_2(-B(M)) > 0.$$

Thus condition (D) is satisfied.

THEOREM 3.2. *Suppose*

(a) $r(x)$ *and* $s(x')$ *are positive, continuous, and satisfy a local Lipschitz condition on* $(-\infty, +\infty)$.

(b) *There exists* $M_1 \geqslant 0$ *such that for* $M \geqslant M_1$,

$$\int_{W_-(M)}^{0} d\sigma/r(h(\sigma, M))\, s(\sigma) > b - a$$

$$\left(\int_0^{W_+(M)} d\sigma/r(h(\sigma, M))\, s(\sigma) > b - a\right),$$

where $h(\sigma, M)$ *is defined on* $(W_-(M), W_+(M))$ *by*

$$-\int_{h(\sigma,M)}^{M} r(\rho)\, d\rho = \int_{\sigma}^{0} \alpha\, d\alpha/s(\alpha).$$

(c) *There exists* $M_2 \geqslant 0$ *such that for* $M \geqslant M_2$,

$$f(t, y, y') > -r(y)\, s(y')$$

on

$$\{(t, y, y') : N(M) \leqslant y' \leqslant 0, y \geqslant h(y', M)\}$$

$$(\{(t, y, y') : 0 \leqslant y' \leqslant P(M), y \geqslant h(y', M)\}),$$

where $P(M)$ *and* $N(M)$ *are defined by*

$$\int_{N(M)}^{0} d\sigma/r(h(\sigma, M))\, s(\sigma) = b - a$$

$$\left(\int_0^{P(M)} d\sigma/r(h(\sigma, M))\, s(\sigma) = b - a\right).$$

(d) *There exists* $M_3 \geqslant 0$ *such that for* $M \geqslant M_3$,

$$h(N(M), M) + g_2(N(M)) > 0$$
$$(h(P(M), M) - g_1(P(M)) > 0).$$

If $y(t)$ *is a solution to* (1.1)–(1.3), *then* $y(t) \leqslant \max[M_1, M_2, M_3, g_1(0), -g_2(0)]$.

Proof. We apply Lemma 2.5.1 with $\psi(x, x') = r(x)\, s(x')$. We have

$$dh/d\sigma = -\sigma/r(h(\sigma, M))\, s(\sigma),$$
$$-r(h(\sigma, M))\, dh/d\sigma = \sigma/s(\sigma),$$
$$-\int_\sigma^0 r(h(\alpha, M))(dh/d\sigma)\, d\alpha = \int_\sigma^0 \alpha\, d\alpha/s(\alpha),$$
$$-\int_{h(\sigma,M)}^M r(\rho)\, d\rho = \int_\sigma^0 \alpha\, d\alpha/s(\alpha).$$

Thus condition (B) is clearly satisfied.

If we note that by the remark following Lemma 2.2, $x'(b, a, M) = N(M)$ and that by Lemma 2.1 $x(b, a, M) = h(N(M), M)$, then it is immediate that conditions (C) and (D) are satisfied.

COROLLARY 3.3. *Suppose*

(i) $r(x)$ *and* $s(x')$ *are positive, continuous, and satisfy a local Lipschitz condition on* $(-\infty, \infty)$ *with* $\int_0^{+\infty} r(\rho)\, d\rho < +\infty$.

(ii) *There exists* $M_2 \geqslant 0$ *such that for* $y \geqslant M_2$, *and* $y' \leqslant 0$ $(y' \geqslant 0)$

$$f(t, y, y') > -r(y)\, s(y').$$

Then there exists $R > 0$ *such that if* $y(t)$ *is a solution to* (1.1)–(1.3), *then* $y(t) \leqslant R$.

Proof. Let $h(\sigma, M)$ be defined on $(W_-(M), W_+(M))$ for $M \geqslant M_2$ as in Theorem 3.2 by

$$-\int_{h(\sigma,M)}^M r(\rho)\, d\rho = \int_\sigma^0 \alpha\, d\alpha/s(\alpha).$$

Note that $W_-(M)$ is decreasing as a function of M. Choose δ such that $W_-(M_2) \leqslant -\delta < 0$. We have

$$\int_{W_-(M)}^0 d\sigma/r(h(\sigma, M))\, s(\sigma) \geqslant \int_{-\delta}^0 d\sigma/r(h(\sigma, M))\, s(\sigma).$$

Making the change of variable

$$u = h(\sigma, M),$$

we obtain

$$\int_{-\delta}^{0} d\sigma/r(h(\sigma, M))\, s(\sigma) = -\int_{h(-\delta, M)}^{M} du/\sigma$$
$$\geqslant (1/\delta)\,[M - h(-\delta, M)].$$

We observe that $h(-\delta, M) \uparrow H(\delta) < +\infty$ as $M \to +\infty$, where

$$\int_{H(\delta)}^{\infty} r(\rho)\, d\rho = \int_{-\delta}^{0} \alpha\, d\alpha/s(\alpha).$$

Thus $[M - h(-\delta, M)]/\delta \to +\infty$ as $M \to +\infty$ and condition (b) of Theorem 3.2 is satisfied.

We also observe that for M sufficiently large

$$N(M) > -\delta.$$

By the same argument as above we have

$$b - a = \int_{N(M)}^{0} d\sigma/r(h(\sigma, M))\, s(\sigma) \geqslant -[M - h(N(M), M)]/N(M).$$

Thus

$$h(N(M), M) \geqslant M + N(M)(b - a)$$
$$\geqslant M - \delta(b - a).$$

If $N(M) \leqslant y' \leqslant 0$ and $y \geqslant h(y', M)$, then

$$y \geqslant h(N(M), M) \geqslant M - \delta(b - a) > M_2 \quad \text{for} \quad M > \overline{M}_2 \equiv M_2 + \delta(b - a).$$

Thus condition (c) is satisfied.

Moreover,

$$h(N(M), M) + g_2(N(M)) \geqslant M - \delta(b - a) + g_2(-\delta) > 0$$

for $M \geqslant M_3 \equiv \delta(b - a) - g_2(-\delta)$ and condition (d) is satisfied.

Corollary 3.4. *Suppose*

(i) $s(x')$ *is positive, continuous, and satisfies a local Lipschitz condition on* $(-\infty, +\infty)$ *with* $\int_{-\infty}^{0} d\sigma/s(\sigma) > b - a(\int_{0}^{+\infty} d\sigma/s(\sigma) > b - a)$.

(ii) *There exists* $M_2 \geqslant 0$ *such that for* $y \geqslant M_2$ *and* $y' \leqslant 0$ $(y' \geqslant 0)$

$$f(t, y, y') > -s(y').$$

There exists R *such that if* $y(t)$ *is a solution to* (1.1)–(1.3), *then* $y(t) \leqslant R$.

Proof. We apply Theorem 3.2 with $r(x) \equiv 1$. We have

$$\int_{h(\sigma,M)}^{M} r(\rho)\, d\rho = -\int_{\sigma}^{0} \alpha\, d\alpha/s(\alpha),$$

$$M - h(\sigma, M) = -\int_{\sigma}^{0} \alpha\, d\alpha/s(\alpha),$$

$$h(\sigma, M) = M + \int_{\sigma}^{0} \alpha\, d\alpha/s(\alpha).$$

Since $W_{-}(M) = -\infty$, condition (b) is satisfied.

Defining $N(M)$ by $\int_{N(M)}^{0} d\sigma/s(\sigma) = b - a$, we observe that $N = N(M)$ is independent of M. If $N(M) \leqslant y' \leqslant 0$ and $y \geqslant h(y', M)$, then $y \geqslant h(N, M) > M_2$ for M sufficiently large. Thus, condition (c) is satisfied. Moreover,

$$h(N(M), M) + g_2(N(M)) \geqslant M + \int_{N}^{0} \alpha\, d\alpha/s(\alpha) + g_2(N) > 0$$

for M sufficiently large. Thus condition (d) is satisfied.

COROLLARY 3.5. *If* $f(t, y, y')$ *is nondecreasing in* y *and satisfies a uniform Lipschitz condition in* y', *then there exists* $R > 0$ *such that if* $y(t)$ *is a solution to* (1.1)–(1.3), *then* $y(t) \leqslant R$.

Proof. We apply Corollary 3.4. We have for $y \geqslant M_2$

$$\begin{aligned} f(t, y, y') - f(t, y, 0) &\geqslant -K\,|\,y'\,|, \\ f(t, y, y') \geqslant \min_{[a,b]} f(t, M_2, 0) - K\,|\,y'\,| &\geqslant -A - K\,|\,y'\,|, \end{aligned} \tag{3.1}$$

where K is the Lipschitz constant. We may take $s(y') = A + K\,|\,y'\,|$. In the remaining theorems of this section we consider the following special case of (1.1)–(1.3):

$$y'' = f(t, y, y'), \tag{3.2}$$

$$y(a) - b_1 y'(a) = a_1, \tag{3.3}$$

$$y(b) + b_2 y'(b) = a_2, \tag{3.4}$$

where $b_1, b_2 \geqslant 0$.

COROLLARY 3.6. *Suppose*

(i) $r(x) = O(|x|^p)$ *as* $|x| \to +\infty$ *and* $s(x') = O(|x'|^q)$ *as* $|x'| \to +\infty$ *with* $p > -1$ *and* $p + q < 1$.

(ii) *There exists* $M_2 \geqslant 0$ *such that for* $y \geqslant M_2$ *and* $y' \leqslant 0$ $(y' \geqslant 0)$

$$f(t, y, y') > -r(y)\, s(y').$$

Then there exists R such that if $y(t)$ *is a solution to* (3.2)–(3.4), *then* $y(t) \leqslant R$.

Proof. Suppose $r(x) < A\,|x|^p$ for $|x| \geqslant \gamma$ and $s(x') \leqslant B\,|x'|^q$ for $|x'| \geqslant \delta$. Let $\bar{r}(x)$ be defined so that

(i) $\bar{r}(x) \geqslant r(x)$ on $(-\infty, +\infty)$,
(ii) $\bar{r}(x) = A\,|x|^p$ for $|x| \geqslant \gamma$,
(iii) $\bar{r}(x) > 0$ on $(-\infty, +\infty)$,
(iv) $\bar{r}(x)$ is continuous and satisfies a local Lipschitz condition on $(-\infty, +\infty)$.

Define $\bar{s}(x')$ similarly. We apply Theorem 3.2 using $\bar{r}(x)$ and $\bar{s}(x')$. Let $h(\sigma, M)$ be defined as in Theorem 3.2 by

$$-\int_{h(\sigma,M)}^{M} \bar{r}(\rho)\, d\rho = \int_{\sigma}^{0} \alpha\, d\alpha/\bar{s}(\alpha).$$

Whenever $h(\sigma, M) \geqslant \gamma$,

$$-\int_{h(\sigma,M)}^{M} \bar{r}(\rho)\, d\rho = -\int_{h(\sigma,M)}^{M} A\rho^p\, d\rho = A[h(\sigma, M)^{p+1} - M^{p+1}]/(p + 1).$$

Moreover, for $\sigma \leqslant -\delta$

$$\begin{aligned}\int_{\sigma}^{0} \alpha\, d\alpha/\bar{s}(\alpha) &= \int_{-\delta}^{0} \alpha\, d\alpha/\bar{s}(\alpha) - \int_{\sigma}^{-\delta} (-\alpha)^{1-q}\, d\alpha/B \\ &= \int_{-\delta}^{0} \alpha\, d\alpha/\bar{s}(\alpha) + [\delta^{2-q} - (-\sigma)^{2-q}]/B(2 - q).\end{aligned}$$

Let $[w_-(M), 0]$ be the interval on which $h(\sigma, M) \geqslant \gamma$. We have

$$h(\sigma, M) = \begin{cases} \left[M^{p+1} + (p + 1)\, A^{-1} \displaystyle\int_{\sigma}^{0} \alpha\, d\alpha/\bar{s}(\alpha)\right]^{1/(p+1)}, & -\delta \leqslant \sigma \leqslant 0 \\ \Big\{M^{p+1} + (p + 1)\, A^{-1} \displaystyle\int_{-\delta}^{0} \alpha\, d\alpha/\bar{s}(\alpha) & \\ \qquad + (p + 1)\, A^{-1}[\delta^{2-q} - (-\sigma)^{2-q}]/B(2 - q)\Big\}^{1/(p+1)}, & \\ & w_-(M) \leqslant \sigma \leqslant -\delta. \end{cases}$$

We observe that for M sufficiently large $w_{-}(M) = -[K_1 M^{p+1} + K_2]^{1/(2-q)}$, where K_1 is a positive constant. We have as in the proof of Corollary 3.3

$$\begin{aligned}\int_{w_{-}(M)}^{0} d\sigma/\bar{r}(h(\sigma, M))\,\bar{s}(\sigma) &\geqslant \int_{w_{-}(M)}^{0} d\sigma/\bar{r}(h(\sigma, M))\,\bar{s}(\sigma)\\ &\geqslant -\int_{\gamma}^{M} du/\sigma\\ &\geqslant -[M-\gamma]/w_{-}(M)\\ &\geqslant [M-\gamma]/[K_1 M^{p+1} + K_2]^{1/(2-q)}.\end{aligned}$$

Note that since $p + q < 1$ and $q < 2$, the expression on the right tends to $+\infty$ as $M \to +\infty$. Thus condition (b) is satisfied.

We observe that $N(M) \geqslant w_{-}(M)$ for M sufficiently large. We have

$$\begin{aligned}b - a = \int_{N(M)}^{0} d\sigma/r(h(\sigma, M))\, s(\sigma) &= -\int_{h(N(M), M)}^{M} du/\sigma\\ \geqslant -[M - h(N(M), M)]/N(M)&\\ \geqslant [M - h(N(M), M)]/[K_1 M^{p+1} + K_2]^{1/(2-q)}.&\end{aligned}$$

Thus

$$h(N(M), M) \geqslant M - [K_1 M^{p+1} + K_2]^{1/(2-q)}\,(b - a).$$

Since $(p + 1)/(2 - q) < 1$, the right-hand side tends to $+\infty$. It follows by arguments similar to those in the proof of Corollary 3.3 that (c) is satisfied.

Moreover,

$$h(N(M), M) + b_2 N(M) \geqslant M - CM^{(p+1)/(2-q)}(b - a) - b_2 CM^{(p+1)/(2-q)} > a_2$$

for M sufficiently large and condition (d) is satisfied.

COROLLARY 3.7. *Suppose*

(i) $r(\rho) = o(\rho)$ *as* $\rho \to +\infty$ *and* $s(\sigma) = O(\sigma)$ *as* $\sigma \to +\infty$.

(ii) *There exists* $M_2 \geqslant 0$ *such that for* $y \geqslant M_2$ *and* $y' \leqslant 0$ $(y' \geqslant 0)$

$$f(t, y, y') > -[r(|\,y\,|) + s(|\,y'\,|)].$$

Then there exists R *such that if* $y(t)$ *is a solution to* (3.2)–(3.4), *then* $y(t) \leqslant R$.

Proof. Suppose $r(\rho) \leqslant A + P(\rho)\rho$, where $P(\rho) \to 0$ as $\rho \to +\infty$. Without loss of generality we may assume that $P(\rho)$ is decreasing, $P(\rho)$ satisfies a local Lipschitz condition, $P(\rho)\rho$ is increasing, and $P(\rho)\rho \to +\infty$

as $\rho \to +\infty$. Suppose $s(\sigma) \leqslant B + C\sigma$. We apply Theorem 3.1 with $\phi(\rho, \sigma) = A + P(\rho)\rho + B + C\sigma$. Clearly, condition (a) is satisfied.

We have

$$\int_0^\infty d\sigma/(A + P(M)M + B + C\sigma) = +\infty,$$

thus condition (b) is satisfied.

If

$$\int_0^{B(M)} d\sigma/(A + P(M)M + B + C\sigma) = b - a,$$

then calculating the integral we have

$$B(M) = (A + B + P(M)M)(\exp C(b - a) - 1)/C.$$

Thus

$$B(M) \leqslant DP(M)M$$

for M sufficiently large.

If $L(\sigma, M) \geqslant 0$, and

$$\int_{L(\sigma,M)}^M (A + P(|u|)|u| + B)\, du = \sigma^2/2,$$

then

$$P(M)\int_{L(\sigma,M)}^M u\, du \leqslant \sigma^2/2,$$

and

$$L(\sigma, M) \geqslant [M^2 - \sigma^2/P(M)]^{1/2}.$$

We then have

$$L(-B(M), M) \geqslant M[1 - D^2P(M)]^{1/2}$$

for M sufficiently large. If $y \geqslant L(y', M)$ and $-B(M) \leqslant y' \leqslant 0$, then $y \geqslant L(-B(M), M) \geqslant M_2$, for M sufficiently large; hence, condition (c) is satisfied.

Moreover,

$$\begin{aligned} &L(-B(M), M) + b_2(-B(M)) - a_2 \\ &\qquad \geqslant M[1 - D^2P(M)]^{1/2} - b_2DP(M)M - a_2 > 0 \end{aligned}$$

for M sufficiently large; hence, condition (d) is satisfied.

THEOREM 3.8. *Suppose there exists $M_2 \geqslant 0$ such that for $y \geqslant M_2$ and $y' \leqslant 0$ ($y' \geqslant 0$), $f(t, y, y') > -A - C|y| - B|y'|$. Suppose*

$$b - a < \Gamma(B, C, b_2)\ (b - a < \Gamma(-B, C, b_1)),$$

where

$$\Gamma(B, C, b_2) \equiv \begin{cases} 2(B^2 - 4C)^{-1/2} \tanh^{-1}[(B^2 - 4C)^{1/2}/(2b_2C + B)], & B^2 - 4C > 0 \\ 2(4C - B^2)^{-1/2} \tan^{-1}[(4C - B^2)^{1/2}/(2b_2C + B)], & B^2 - 4C < 0 \\ 4/(2B + b_2B^2), \quad B^2 - 4C = 0. \end{cases}$$

There exists R such that if $y(t)$ is a solution to (3.2)–(3.4), *then $y(t) \leqslant R$.*

Proof. We apply Lemma 2.5.1 with $\psi(x, x') = A + C|x| + B|x'|$. Clearly condition (A) is satisfied. Let $x(t, t_0, M)$ denote the unique solution to

$$x'' = -A - C|x| - B|x'|,$$
$$x(t_0) = M,$$
$$x'(t_0) = 0.$$

As long as $x(t, t_0, M)$ is positive for $t \geqslant t_0$, $x(t, t_0, M)$ is the solution to

$$x'' - Bx' + Cx = -A,$$
$$x(t_0) = M,$$
$$x'(t_0) = 0.$$

We give the proof only in the case $B^2 - 4C > 0$. The other cases are similar. The general solution to the equation is

$$\phi(t) = e^{B(t-t_0)/2}[c_1 \cosh (B^2 - 4C)^{\frac{1}{2}}(t - t_0)/2 + c_2 \sinh (B^2 - 4C)^{\frac{1}{2}}(t - t_0)/2] - (A/C).$$

Let $\alpha = (B^2 - 4C)^{\frac{1}{2}}/2$. Then

$$\phi'(t) = (B/2)\, e^{B(t-t_0)/2}[c_1 \cosh \alpha(t - t_0) + c_2 \sinh \alpha(t - t_0)] + e^{B(t-t_0)/2}\alpha[c_1 \sinh \alpha(t - t_0) + c_2 \cosh \alpha(t - t_0)].$$

We must have

$$\phi'(t_0) = C_1B/2 + \alpha C_2 = 0,$$
$$\phi(t_0) = C_1 - A/C = M.$$

Thus $C_1 = M + A/C$ and $C_2 = -B(M + A/C)/2\alpha$. Thus

$$x(t, t_0, M) = e^{B(t-t_0)/2}(M + A/C) \times [\cosh \alpha(t - t_0) - (B/2\alpha) \sinh \alpha(t - t_0)] - A/C.$$

$$\begin{aligned} x'(t, t_0, M) = {} & (B/2)(M + A/C)\, e^{B(t-t_0)/2} \\ & \times [\cosh \alpha(t - t_0) - (B/2\alpha) \sinh \alpha(t - t_0)] \\ & + (M + A/C)\, e^{B(t-t_0)/2} \\ & \times \alpha[\sinh \alpha(t - t_0) - (B/2\alpha) \cosh \alpha(t - t_0)]. \end{aligned} \tag{3.5}$$

Note that if

$$\cosh \alpha(t - t_0) - (B/2\alpha) \sinh \alpha(t - t_0) > 0$$

on $[t_0, b]$, then $x(t, t_0, M)$ is represented by (3.5) on $[t_0, b]$. This is guaranteed if

$$\tanh \alpha(t - t_0) < 2\alpha/B \text{ on } [t_0, b]$$

or

$$\tanh \alpha(b - a) < 2\alpha/B$$

or

$$b - a < \alpha^{-1} \tanh^{-1}(2\alpha/B). \tag{3.6}$$

Note that on $x'(b, a, M) \leqslant y' \leqslant 0$, $h(y', M) \geqslant h(x'(b, a, M), M) = x(b, a, M)$. If M is sufficiently large and (3.6) holds, then $x(b, a, M) \geqslant M_2$. Thus (C) is satisfied.

We have

$$\begin{aligned} & x(b, a, M) + b_2 x'(b, a, M) \\ & \quad = e^{B(b-a)/2}(M + A/C)\{[\cosh \alpha(b - a) - (B/2\alpha) \sinh \alpha(b - a)] \\ & \qquad + b_2(B/2)[\cosh \alpha(b - a) - (B/2\alpha) \sinh \alpha(b - a)] \\ & \qquad + b_2\alpha[\sinh \alpha(b - a) - (B/2\alpha) \cosh \alpha(b - a)]\} - (A/C). \end{aligned}$$

The term in brackets may be written

$$\cosh \alpha(b - a) + [\alpha b_2 - b_2(B^2/4\alpha) - (B/2\alpha)] \sinh \alpha(b - a)$$

or

$$\cosh \alpha(b - a) - [(2b_2C + B)/2\alpha] \sinh \alpha(b - a).$$

The latter expression is positive if

$$\tanh \alpha(b - a) < 2\alpha/(2b_2C + B)$$

or

$$b - a < \alpha^{-1} \tanh^{-1} 2\alpha/(2b_2C + B). \tag{3.7}$$

Thus condition (D) is satisfied.

By the remark following Lemma 2.2, since we have the existence of $x(t, a, M)$ on $[a, b]$ condition (B) is satisfied, and the proof in this case is complete.

Remark. If $f(t, y, y') > -A - B\,|\,y\,| - C\,|\,y'\,|$ for $y \geqslant M_2$ and all y', then it is sufficient that

$$b - a < 2\min[\Gamma(B, C, b_2), \Gamma(-B, C, b_1)].$$

Remark. Each of the theorems and corollaries of this section has an analogue giving sufficient conditions for the existence of a lower bound. These analogues are obtained from Lemma 2.5.2.

4. Existence of Upper Solutions, Lower Solutions, and Solutions

A function $\beta(t)$ is an *upper solution* for problem (1.1)–(1.3) if $\beta(t) \in C^2[a, b]$, $\beta''(t) \leqslant f(t, \beta(t), \beta'(t))$ on $[a, b]$, $\beta(a) - g_1(\beta'(a)) \geqslant 0$, and $\beta(b) + g_2(\beta'(b)) \geqslant 0$. If the inequalities are replaced by strict inequalities, then $\beta(t)$ is a *strict upper solution. Lower solution* and *strict lower solution* are defined similarly with the inequalities reversed.

Theorem 4.1. *If* (A)–(D) *are satisfied, then* $\beta(t) = x(t, a, M)\,(x(t, b, M))$ *is a strict upper solution for* $M > \max[M_1, M_2, M_3, g_1(0), -g_2(0)]$. *If* (A), (B′), (C′), *and* (D′) *are satisfied, then* $\alpha(t) \equiv x(t, a, M)\,(x(t, b, M))$ *is a strict lower solution for* $M < \min[-M_1, -M_2, -M_3, g_1(0), -g_2(0)]$.

Proof. We consider only the upper solution case. By Lemma 2.2, $x(t, a, M) \in C^2[a, b]$. By condition (C), since $x(t, a, M) = h(x'(t, a, M), M)$, and $x'(t, a, M) \geqslant x'(b, a, M)$,

$$\begin{aligned} x''(t, a, M) &= -\psi(x(t, a, M), x'(t, a, M)) \\ &< f(t, x(t, a, M), x'(t, a, M)) \end{aligned}$$

on $[a, b]$. We also have

$$x(a, a, M) - g_1(x'(a, a, M)) = M - g_1(0) > 0,$$

and by condition (D)

$$x(b, a, M) + g_2(x'(b, a, M)) > 0.$$

It follows from Theorem 4.1 that under the hypotheses of any of the theorems and corollaries of Section 3, there exists an upper solution to problem (1.1)–(1.3). Under analogous hypotheses there exists a lower solution. A function $f(t, y, y')$ is said to satisfy a *Nagumo condition* if for any $M > 0$ there exists a positive continuous function $\phi_M(p)$ defined on $[0, \infty)$ such that

$$\int_{2M/(b-a)}^{\infty} p\, dp/\phi_M(p) > 2M$$

and for $|y| \leqslant M$

$$|f(t, y, y')| \leqslant \phi_M(|y'|).$$

THEOREM 4.2. *If there exist upper and lower solutions $\beta(t)$ and $\alpha(t)$ for problem* (1.1)–(1.3) *such that $\alpha(t) \leqslant \beta(t)$ on $[a, b]$ and $f(t, y, y')$ satisfies a Nagumo condition, then problem* (1.1)–(1.3) *has a solution $y(t)$ such that $\alpha(t) \leqslant y(t) \leqslant \beta(t)$.*

Proof. More general versions of this theorem may be found in [4] and [5].

Using Theorem 4.1 in conjunction with Theorem 4.2 we may now obtain an existence theorem corresponding to each of the results in Section 3. The existence theorem corresponding to Corollary 3.4 may be found in [1]. The theorem corresponding to Corollary 3.5 is the existence portion of a theorem in [7]. The theorem corresponding to Theorem 3.8 is a small improvement of the existence portion of a result of Waltman [8]. As illustrations we state two of the remaining theorems, sacrificing some generality in order to simplify hypotheses.

THEOREM 4.3. *Suppose*

(i) *$\phi(\rho, \sigma)$ is positive, continuous, satisfies a local Lipschitz condition in ρ and σ, and is nondecreasing in ρ and σ on $[0, \infty) \times [0, \infty)$.*

(ii) *For all M*

$$\int_0^{\infty} d\sigma/\phi(|M|, \sigma) > b - a.$$

(iii) *There exists $M_2 \geqslant 0$ such that for $|M| \geqslant M_2$,*

$$yf(t, y, y') > -|y|\phi(|y|, |y'|)$$

on

$$\{(t, y, y') : -B(M) \leqslant y' \leqslant 0, y^2 \geqslant |y| L(y', M)\}$$
$$(\{(t, y, y') : 0 \leqslant y' \leqslant B(M), y^2 \geqslant |y| L(y', M)\}),$$

where

$$\int_0^{B(M)} d\sigma/\phi(|M|, \sigma) = b - a$$

and

$$\int_{L(\sigma,M)}^{|M|} \phi(|u|, 0)\, du = \sigma^2/2.$$

(iv) *There exists M_3 such that for $|M| \geqslant M_3$*

$$L(B(M), M) > \max[|g_2(-B(M))|, |g_1(-B(M))|]$$
$$(L(B(M), M) > \max[|g_1(B(M))|, |g_2(B(M))|]).$$

(v) *$f(t, y, y')$ satisfies a Nagumo condition.*

Then (1.1)–(1.3) *has a solution $y(t)$ with*

$$|y(t)| \leqslant \max[M_2, M_3, |g_1(0)|, |g_2(0)|].$$

Proof. By the proof of Theorem 3.1, (A)–(D) are satisfied; hence, by Theorem 4.1 $x(t, a, M)$ is an upper solution for $M > 0$ sufficiently large. Analogous arguments may be used to show that $x(t, b, -M)$ is a lower solution for $M > 0$ sufficiently large. It can easily be shown that $x(t, a, M) \geqslant x(t, b, -M)$ on $[a, b]$ for M sufficiently large. Theorem 4.2 may then be applied to complete the proof.

THEOREM 4.4. *Suppose*

$$|f(t, y, y')| < r(y)\, s(y'),$$

where $r(y) = O(|y|^p)$ as $|y| \to +\infty$ and $s(y') = o(|y'|^q)$ as $|y'| \to +\infty$ with $p > -1$ and $p + q < 1$. Then problem (3.2)–(3.4) *has a solution.*

Proof. Note that since $p > -1$ and $p + q < 1$ we have $q < 2$. Thus it can easily be verified that $f(t, y, y')$ satisfies a Nagumo condition. The theorem then follows from the proofs of Theorem 3.2 and Corollary 3.6, Theorem 4.1, and Theorem 4.2.

Remark. The existence theorems described above may also be obtained by means of the Leray–Schauder fixed-point theorem using the *a priori* bounds of Section 2 and a bound on derivatives of solutions obtained from the Nagumo condition. This is the approach in [1].

REFERENCES

1. R. GAINES, *A priori* bounds for solutions to nonlinear two-point boundary value problems, *Applicable Anal.* to appear.
2. JU. A. KLOKOV, A certain two-point problem for ordinary differential equations, Latvian Math. Yearbook, 3, pp. 177–200, Izdat. "Zinatne," Riga, 1968.

3. L. Jackson, Subfunctions and second order ordinary differential inequalities, *Advances in Math.* **2** (1968), 307–363.
4. L. Erbe, Nonlinear boundary value problems for second order differential equations, *J. Differential Equations* **7** (1970), 459–472.
5. J. Bebernes and R. Fraker, *A priori* bounds for boundary sets, *Proc. Amer. Math. Soc.*, to appear.
6. K. Schmitt, A nonlinear boundary value problem, *J. Differential Equations* **7** (1970), 527–537.
7. J. Bebernes and R. Gaines, A generalized two-point boundary value problem, *Proc. Amer. Math. Soc.* **19** (1968), 749–754.
8. P. Waltman, A nonlinear boundary value problem, *J. Differential Equations* **4** (1968), 597–603.
9. P. Hartman, "Ordinary Differential Equations," Wiley, New York, 1964.
10. K. Schmitt, On the global existence of solutions of second order ordinary differential equations, *J. Differential Equations* **5** (1969), 476–482.